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Angular momentum transport by internal waves in the solar interior

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Abstract. The internal gravity waves of low frequency which are emitted at the base of the solar convection zone are able to extract angular momentum from the radiative interior. We evaluate this transport with some simplifying assumptions: we ignore the Coriolis force, approximate the spectrum of turbulent convection by the Kolmogorov law, and couple this turbulence to the internal waves through their pressure fluctuations, following Press (1981) and García López & Spruit (1991). The local frequency of an internal wave varies with depth in a differentially rotating star, and it can vanish at some location, thus leading to enhanced damping (Goldreich & Nicholson 1989). It is this dissipation mechanism only that we take into account in the exchange of momentum between waves and stellar rotation. The flux of angular momentum is then an implicit function of depth, involving the local rotation rate and an integral representing the cumulative effect of radiative dissipation. We find that the efficiency of this transport process is rather high: it operates on a timescale of 10^7 years, and is probably responsible for the flat rotation profile which has been detected through helioseismology.

Key words: Hydrodynamics, turbulence; Sun: interior, rotation; Stars: interiors, rotation.

1. Introduction

Although it is well known that waves do carry momentum, this process has received little attention so far in stellar physics.The transport of angular momentum by internal waves (also called gravity waves) has been studied in the context of tidal braking involving massive binary stars (Zahn 1975a; Goldreich & Nicholson 1989), but only recently has it been invoked as a mechanism which could shape the rotation profile in the Sun (Zahn 1990; Schatzman 1993). In contrast, the importance of the momentum transport by such waves has been recognized long ago in atmospheric sciences (cf. Bretherton 1969a,b): this phenomenon is responsible in particular for the so-called clear air turbulence.

The purpose of the present paper is to assess the efficiency of angular momentum transport in a solar-type star, where such waves are generated by the turbulent motions of the convection zone. In this first approach we shall make several simplifying assumptions, fully aware that the outcome must be considered as a crude approximation. The results may be easily transposed to massive stars, where these waves are emitted by the convective core.

The reason for examining the role of such waves in the Sun is that the other mechanisms which have been analyzed so far seem unable to achieve the flat rotation profile revealed by helioseismology (cf. Brown et al. 1989). Both rotation-induced turbulent diffusion (Endal & Sofia 1978; Pinsonneault et al. 1989) and wind-driven meridian circulation (Zahn 1992) fail to extract sufficient angular momentum from the radiative interior (Chaboyer et al. 1995; Matias & Zahn 1996). Magnetic torquing looks at first sight more promising, but the field lines anchored in the differentially rotating convection zone would probably enforce non uniform rotation below (cf. Charbonneau & MacGregor 1993), and this is not observed.

We begin by recalling the main properties of the internal waves (§2), and calculate the flux of angular momentum carried by a monochromatic wave (§3). We then deduce the energy spectrum of those waves from their coupling with the turbulent motions at the base of the convection zone (§4) and integrate the angular momentum flux over the whole spectrum (§5). We finally derive an estimate for the efficiency of the angular momentum transport by such waves in the Sun (§6) and end by some concluding remarks.

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2. Properties of internal waves

The properties of internal waves propagating in stellar interiors have been described by Press (1981) and by Goldreich & Nicholson (1989). Let us recall the main results, and adapt them to a differentially rotating star. We use the spherical coordinates (r, θ, ϕ) suited for this problem; $\boldsymbol{e}_z$ is the unit vector on the rotation axis, and $\boldsymbol{e}_\phi$ that in the azimuthal direction. We further assume that the angular velocity Ω depends only on depth, because differential rotation in latitude is severely limited through hydrodynamical instabilities (Zahn 1975b, 1992). The velocity field with respect to an inertial frame is then

$$\boldsymbol{U}(r,\theta,\phi,t) = \Omega(r)\boldsymbol{e}_z \times \boldsymbol{r} + \boldsymbol{u}(r,\theta,\phi,t), \tag{1}$$

with $\boldsymbol{u}$ being the velocity associated with the wave. The equation of motion reads

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} &+ \left\{2\Omega\boldsymbol{e}_z \times \boldsymbol{u} + \boldsymbol{e}_\phi\, r\sin\theta\, \boldsymbol{u}\cdot\boldsymbol{\nabla}\Omega\right\} \\ &= -\frac{1}{\rho}\boldsymbol{\nabla}P' + \frac{\rho'}{\rho}\boldsymbol{g},\end{aligned} \tag{2}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t} = \left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi}\right), \tag{3}$$

and the usual notations for the gravity $\boldsymbol{g}$, and for the Eulerian perturbations of pressure (P') and density (ρ'). We have simplified this equation by neglecting the fluctuations of the gravitational potential: the Cowling approximation is amply justified here, considering the high radial order of the waves. Viscosity also has been ignored. We add the continuity equation

$$\frac{\mathrm{d}\rho'}{\mathrm{d}t} + \boldsymbol{\nabla}\cdot\rho\boldsymbol{u} = 0, \tag{4}$$

and the energy equation, which in the adiabatic limit reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho'}{\rho} - \frac{1}{\Gamma_1}\frac{P'}{P}\right) + \left[\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} - \frac{1}{\Gamma_1}\frac{\mathrm{d}\ln P}{\mathrm{d}r}\right]u_r = 0, \tag{5}$$

Γ_1 being the adiabatic exponent.

We now proceed with a further simplification, which admittedly is much less justified. Neglecting the terms in curly brackets in the equation of motion (2), we treat the waves as if they were pure gravity waves which are not modified by the Coriolis acceleration, but just feel the entrainment by the differential rotation.

Then the differential system above is very similar to that governing the internal waves in a non-rotating star; the only difference lies in the derivative with respect to time, which is replaced here by (3). The solutions are separable in spherical functions and time, as shown for the vertical component of the velocity:

$$u_r(r,\theta,\phi,t) = u_v(r)P_\ell^m(\cos\theta)\exp i[\sigma t - m(\phi-\Omega t)], \tag{6}$$

where

$$\sigma(r) = \sigma_0 - m\Omega(r), \tag{7}$$

with σ_0 being the frequency in the inertial frame. In the low frequency range of the internal waves which are considered here, the function $\Psi(r) = \rho^{\frac{1}{2}}r^2u_v$ obeys the second order equation (cf. Press 1981):

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} + \left(\frac{N^2}{\sigma^2} - 1\right)\frac{\ell(\ell+1)}{r^2}\Psi = 0\,. \tag{8}$$

The buoyancy (or Brunt-Väisälä) frequency N is given by

$$N^2 = N_t^2 + N_\mu^2 = \frac{g\delta}{H_P}(\nabla_{\mathrm{ad}} - \nabla) + \frac{g\varphi}{H_P}\nabla_\mu \tag{9}$$

where we have used the classical notations of stellar structure theory: H_P is the pressure scale height $P/\rho g$, $\nabla = \partial\ln T/\partial\ln P$ the logarithmic temperature gradient, $\delta = -(\partial\ln\rho/\partial\ln T)_{P,\mu}$ and $\varphi = (\partial\ln\rho/\partial\ln\mu)_{P,T}$. We allow for a molecular weight gradient $\nabla_\mu = \mathrm{d}\ln\mu/\mathrm{d}\ln P$, whose contribution may be important in some cases. Let us also recall that, due to convective penetration, N starts at a finite value N_c at the top of the radiation zone (Zahn 1991).

We introduce the vertical wavenumber k_v:

$$k_v^2 = \left(\frac{N^2}{\sigma^2} - 1\right)\frac{\ell(\ell+1)}{r^2} \tag{10}$$

and observe that $rk_v \gg 1$ for $\sigma \ll N$. Therefore the differential equation (8) may be solved by the WKB method, which yields the following result:

$$\begin{aligned}u_r &= \mathrm{C}\, r^{-\frac{3}{2}}\rho^{-\frac{1}{2}}\left(\frac{N^2}{\sigma^2} - 1\right)^{-\frac{1}{4}}P_\ell^m(\cos\theta) \\ &\times\ \cos\left(\sigma t - m(\phi - \Omega t) - \int_r^{r_c} k_v\mathrm{d}r\right),\end{aligned} \tag{11}$$

with r_c designating the base of the convection zone and the constant C fixing the amplitude (cf. Press 1981). It describes a "monochromatic" wave of spherical order ℓ and local frequency σ, which propagates[1] with the phase velocity $(-\sigma/k_v,\ 0,\ r\sin\theta\,\sigma/m)$, m being the azimuthal order; its vertical group velocity is given by

$$V_g = -\frac{\mathrm{d}\sigma}{\mathrm{d}k_v} = \frac{\sigma}{k_v}\,\frac{N^2 - \sigma^2}{N^2}. \tag{12}$$

Making use of the continuity equation (4), we obtain similar expressions for the horizontal components of the velocity. In particular

$$\begin{aligned}u_\phi &= \mathrm{C}\, m\,\frac{rk_v}{\ell(\ell+1)}\, r^{-\frac{3}{2}}\rho^{-\frac{1}{2}}\left(\frac{N^2}{\sigma^2} - 1\right)^{-\frac{1}{4}}\frac{P_\ell^m(\cos\theta)}{\sin\theta} \\ &\times\ \cos\left(\sigma t - m(\phi - \Omega t) - \int_r^{r_c} k_v\mathrm{d}r\right) \\ &= m\,\frac{rk_v}{\ell(\ell+1)}\,\frac{u_r}{\sin\theta}\,.\end{aligned} \tag{13}$$

[1] The most general solution would include a stationary wave.

3. Fluxes associated with a monochromatic wave

The horizontal average of the kinetic energy density is easily deduced from the WKB solution above:

$$\begin{aligned}\frac{1}{2}\,\rho < u^2 > \;&\equiv\; \frac{1}{4\pi}\iint \frac{1}{2}\,\rho(u_r^2+u_\theta^2+u_\phi^2)\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &=\; \frac{1}{2}\,\frac{N^2}{\sigma^2}\frac{1}{4\pi}\iint \rho\,u_r^2\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \qquad (14) \\ &\equiv\; \frac{1}{2}\,\frac{N^2}{\sigma^2}\,\rho\,< u_r^2 > .\end{aligned}$$

Multiplying by the group velocity (12), we get the average flux of kinetic energy transported by a traveling wave:

$$\begin{aligned}\mathcal{F}_K \;&=\; \frac{1}{2}\,\rho < u^2 > \frac{\sigma}{k_v}\frac{N^2-\sigma^2}{N^2} \\ &=\; \mathrm{sign}(k_v)\,\frac{1}{2}\,\rho < u^2 > \frac{\sigma^2}{N^2}\,\frac{(N^2-\sigma^2)^{\frac{1}{2}}}{k_h}, \qquad (15)\end{aligned}$$

with $k_h = \sqrt{\ell(\ell+1)}/r$ being the horizontal wavenumber.

Likewise we evaluate the mean flux of angular momentum carried by the wave:

$$\begin{aligned}\mathcal{F}_J \;&=\; \frac{1}{4\pi}\iint \rho\,r\sin\theta\,u_\phi u_r\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &=\; m\,\rho r\,\frac{rk_v}{\ell(\ell+1)}\,\frac{1}{4\pi}\iint u_r^2\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \qquad (16) \\ &=\; 2\,\frac{m}{\sigma}\,\mathcal{F}_K, \qquad (17)\end{aligned}$$

where we have used (10), (13) and (15). Inserting (11) into (16), we verify that the angular momentum is conserved in this adiabatic limit[2]:

$$4\pi r^2 \mathcal{F}_J(r) \equiv \mathcal{L}_J = \mathrm{cst}. \qquad (18)$$

In contrast, the kinetic energy of the wave varies with depth, since the local frequency σ depends on the rotation rate $\Omega(r)$, but the conservation of energy is ensured in the inertial frame, as was explained by Bretherton (1969a) in the plane-parallel case.

When radiative damping is taken into account in the quasi-adiabatic limit, the wave amplitude is multiplied by an attenuation factor $\exp(-\tau/2)$, where τ, which is similar to an optical depth, is the integral

$$\tau(r) = [\ell(\ell+1)]^{\frac{3}{2}} \int_r^{r_c} K\,\frac{N N_t^2}{\sigma^4}\left(\frac{N^2}{N^2-\sigma^2}\right)^{\frac{1}{2}}\frac{\mathrm{d}r}{r^3}, \qquad (19)$$

with K being the thermal diffusivity (see Appendix B). Then the angular momentum luminosity is no longer conserved, but decreases as

$$\mathcal{L}_J(r) = \mathcal{L}_J(r_c)\,\exp[-\tau(r)], \qquad (20)$$

which means that some angular momentum will be extracted from the radiative interior.

[2] This condition is not fulfilled in Schatzman (1993).

4. Spectral distribution

From now on, we focus only on those waves which propagate towards the radiative interior and thus carry energy upwards ($k_v > 0$). As we have seen above in (15), the average kinetic energy flux transported by such a wave, at the top of the radiation zone ($r = r_c$), is given by

$$\mathcal{F}_K(r_c) = \frac{1}{2}\,\rho\,\frac{(N_c^2-\omega^2)^{\frac{1}{2}}}{k_h}\,\frac{\omega^2}{N_c^2}\,\overline{u}^2, \qquad (21)$$

where $\overline{u}(\omega, k_h)$ is the r.m.s. of the wave velocity over the horizontal surface, N_c the buoyancy frequency, and ω the frequency of the wave in the frame rotating with the angular velocity Ω_c of the convection zone (Press 1981; García López & Spruit 1991).

We now turn to the evaluation of the flux carried by the whole spectrum of waves generated by the convective motions. In all likelihood the excitation of these internal waves takes place very close to the base of the convection zone. There are two reasons for that. First the gravity waves do not penetrate far into the convection zone, since they are evanescent there with the lapse rate k_h. And second the interface is rather sharp, with a jump in the buoyancy frequency and plumes penetrating into the radiative interior, which favors the generation of waves as one observes both in the laboratory (Townsend 1958) and in numerical simulations (Hurlburt et al. 1986, 1994).

To obtain a crude estimate of the wave energy, we follow closely García López & Spruit (1991). We match the pressure fluctuation in the wave with that of the turbulent convection, thus

$$\overline{u}^2(\omega) = v^2(\omega), \qquad (22)$$

where $v(\omega)$ designates the r.m.s. convective velocity at the frequency ω. We further take into account that convective eddies of wavenumber $k(\omega)$ generate also waves at lower (horizontal) wavenumber k_h. Summing the eddies which participate in the stochastic excitation of a wave of wavenumber k_h, we get

$$\overline{u}^2(\omega, k_h) = \left[\frac{k_h}{k(\omega)}\right]^2 v^2(\omega) \equiv 2\int_0^{k_h} v^2(\omega)\left[\frac{k_h}{k(\omega)}\right]^2 \frac{\mathrm{d}k_h}{k_h}. \qquad (23)$$

Next we assume that the kinetic energy spectrum of the convective motions is represented to first approximation by the Kolmogorov law:

$$v^2(\omega) = \int_\omega^\infty v_c^2 \left[\frac{\omega}{\omega_c}\right]^{-1} \frac{\mathrm{d}\omega}{\omega} \quad \text{with} \quad \omega \geq \omega_c, \qquad (24)$$

where v_c and ω_c (and k_c below) characterize the largest convective eddies. We invoke once more Kolmogorov's law to replace $k^2(\omega)$ in (23) by $k_c^2(\omega/\omega_c)^3$, and reach the following expression for the flux of kinetic energy at the top of the radiative interior:

$$\mathcal{F}_K(r_c) = \rho_c v_c^3 \frac{\omega_c}{N_c^2}\int_{\omega_c}^{N_c} \frac{\mathrm{d}\omega}{\omega}\,(N_c^2-\omega^2)^{\frac{1}{2}}\left[\frac{\omega}{\omega_c}\right]^{-2}\int_0^\ell \frac{\mathrm{d}\ell}{\ell_c}. \qquad (25)$$

From now on we use the spherical harmonic number $\ell = rk_h$, and pretend that it varies continuously from 0 to its upper bound ℓ_u (the convective scale):

$$0 < \ell < \ell_u \quad \text{with } \ell_u = \ell_c \left(\frac{\omega}{\omega_c}\right)^{\frac{3}{2}}. \tag{26}$$

In the limit $\omega_c \ll N_c$ the flux integrated over the whole spectrum amounts to

$$\mathcal{F}_K(r_c) = 2\,\rho v_c^3 \left(\frac{\omega_c}{N_c}\right), \tag{27}$$

a result which is similar to that given by Press for his monochromatic flux [cf. his eq. (90)]. Note however that our expression (25) for $\mathcal{F}_K$ vanishes when $\omega_c \to N_c$, whereas his is singular there.

To proceed with the evaluation of the angular momentum flux, we need to know how the spectral energy of the internal waves is distributed over the azimuthal order m, for given ℓ. For simplicity, we shall assume that this distribution is uniform, although it is quite possible that the Coriolis force, which we have neglected here, causes an unbalance between prograde and retrograde waves, hence between positive and negative m. With our simplifying assumption

$$\begin{aligned}\mathcal{F}_K(r_c) \;&=\; \rho_c v_c^3 \frac{\omega_c}{N_c^2} \qquad\qquad (28)\\ &\times \int_{\omega_c}^{N_c} \frac{\mathrm{d}\omega}{\omega}\,(N_c^2-\omega^2)^{\frac{1}{2}} \left[\frac{\omega}{\omega_c}\right]^{-2} \int_0^{\ell_u} \frac{\mathrm{d}\ell}{\ell_c} \int_{-\ell}^{\ell} \frac{\mathrm{d}m}{2\ell}.\end{aligned}$$

Referring back to (17) and (20), we obtain the following expression for the luminosity of angular momentum integrated over the whole wave spectrum:

$$\begin{aligned}\mathcal{L}_J(r) &= 4\pi r^2 \frac{\rho_c v_c^3}{N_c \ell_c} \qquad\qquad (29)\\ &\times \int_{\omega_c}^{N_c} \frac{\mathrm{d}\omega}{\omega} \left(1-\frac{\omega^2}{N_c^2}\right)^{\frac{1}{2}} \left[\frac{\omega}{\omega_c}\right]^{-3} \int_0^{\ell_u} \frac{\mathrm{d}\ell}{\ell} \int_{-\ell}^{\ell} \exp(-\tau)\, m\,\mathrm{d}m.\end{aligned}$$

In a stationary regime, one would have to subtract from this luminosity that, of opposite sign, which is carried by the waves reflected at the center, including the appropriate damping.

5. The angular momentum flux

In a non-rotating star, there is no net flux of angular momentum, since two waves propagating in the same direction but which are of opposite azimuthal order m have the same amplitude; therefore they carry (and deposit) equal amounts of angular momentum, which cancel each other because they are of opposite sign.

The situation is different in a rotating star, because the Coriolis force may introduce a bias between waves of opposite m, and this could generate differential rotation as the waves undergo radiative damping. We do not consider this possibility here, and assume rather that differential rotation exists for another reason, such as mass loss or meridian circulation. In this case, waves of opposite azimuthal order experience different damping, and the net effect will be an extraction of angular momentum from the radiation zone. When the damping is slight, the associated flux of angular momentum will be small, and we shall neglect it in this first approach. Instead, we concentrate on those waves which are completely damped because their local frequency vanishes at some critical depth. A similar damping process plays an important role in the tidal synchronization of massive binaries, as was pointed out by Goldreich & Nicholson (1989).

The local frequency may be written

$$\sigma(r) = \omega - m\,[\Omega(r) - \Omega_c] \equiv \omega - m\,\Delta\Omega(r) \tag{30}$$

with ω being the frequency of the monochromatic wave when it is emitted at the base of the convection zone, which rotates at the angular velocity Ω_c. If the rotation speed $\Omega(r)$ increases with depth, σ will have a node at some location $r^*(\omega, m)$ for sufficiently large (and positive) m. There $k_v \to \infty$ and radiative damping will increase dramatically – formally $\tau \to \infty$, but the quasi-adiabatic approximation leading to (19) is no longer valid. Thus at the critical layer $r = r^*$ this wave will deposit whatever remains from its (negative) angular momentum. The result is similar if the rotation rate decreases with depth, but it is then of opposite sign; in each case radiative dissipation acts to reduce the existing differential rotation.

From now on, we shall consider only those waves which experience little damping before they reach the critical level r^* where $\sigma \to 0$; according to (19), their initial frequency ω is such that $\tau \lesssim 1$, or

$$\omega^4 \gtrsim I(r^*)\,\ell^3 \quad \text{where } I(r) = \int_r^{r_c} K\,N\,N_t^2\,\frac{\mathrm{d}r}{r^3}. \tag{31}$$

A wave which does not meet this condition is damped before, but its contribution to the transport of angular momentum is compensated by that of its partner of opposite m and we shall ignore it in this first approach.

A final, admittedly drastic simplification will enable us to calculate the triple integral (29): we shall approximate the damping exponential by the step function $\mathcal{H}(r-r^*)$. We thus take $\exp[-\tau(r)] = 1$ for $\omega > m\,\Delta\Omega$ and $\exp[-\tau(r)] = 0$ for $\omega < m\,\Delta\Omega$. In other words, $\exp[-\tau]$ will be removed from (29), and the integration domain in m will be delimited by

$$0 < m < \min[\ell, \omega/\Delta\Omega]. \tag{32}$$

Let us recall the other limits; according to (31) and (26)

$$0 < \ell < \min[(\omega^4/I)^{\frac{1}{3}}, \ell_c(\omega/\omega_c)^{\frac{3}{2}}] \tag{33}$$

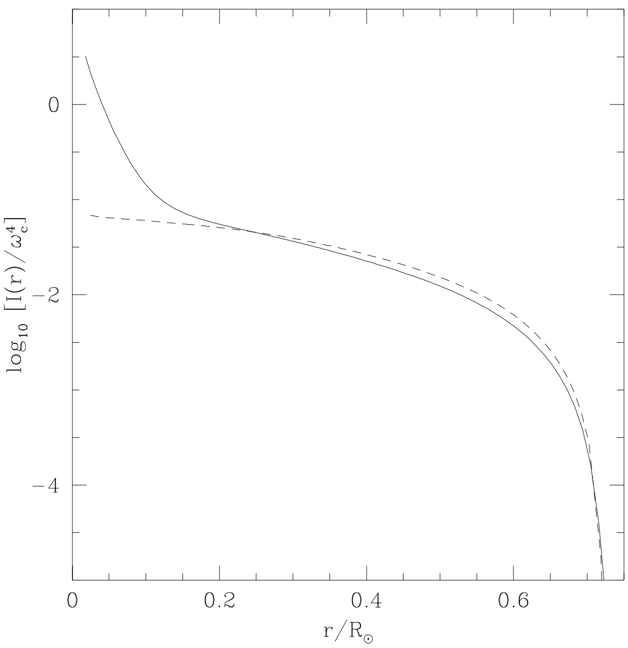

Fig. 1. Variation with depth of the damping integral defined in (31), scaled by the fourth power of the convective turnover frequency: I/ω_c^4. The dashed line refers to the Sun at 200 Myr, the continuous line to the present Sun. The radial cooordinate is normalized by the actual, time-dependent radius of the Sun.

and for the frequency we have

$$\omega_c < \omega < N_c. \tag{34}$$

The integration can now be performed, and its results are reported in Appendix A. The angular momentum flux is an implicit function of r, through $\Delta\Omega(r)$ and $I(r)$, whose variation with depth is displayed in Fig. 1. Its general expression is rather intricate, but it simplifies in some parameter domains. For instance, let us take the interval in I which is the most relevant for the solar interior, namely where

$$\ell_c^{-3} < \left(\frac{I}{{\omega_c}^4}\right) < 1 \tag{35}$$

and that in $\Delta\Omega(r)$ defined by

$$\frac{\omega_c}{N_c}\left(\frac{I}{{\omega_c}^4}\right) < \left(\frac{\Delta\Omega}{\omega_c}\right)^3 < \left(\frac{I}{{\omega_c}^4}\right) \tag{36}$$

(for the physical meaning of these limits we refer the reader to Appendix A.) In this domain the flux depends linearly on the differential rotation:

$$\mathcal{F}_J = \frac{\rho_c v_c^3}{N_c \ell_c}\left[\frac{3}{4}\left(\frac{{\omega_c}^4}{I}\right)^{\frac{2}{3}} - \frac{1}{3}\left(\frac{{\omega_c}^4}{I}\right)\frac{\Delta\Omega}{\omega_c}\right]. \tag{37}$$

The term independent of $\Delta\Omega$ represents the contribution of waves which do not meet the singularity at $\sigma = 0$, and consistent with the approximation made above, we shall ignore it. We thus write the result as

$$\mathcal{L}_J(r) = \mathcal{L}_J(r_c) - \frac{4\pi r^2}{3}\frac{\rho_c v_c^3}{N_c \ell_c}\left(\frac{{\omega_c}^4}{I}\right)\frac{\Delta\Omega}{\omega_c}. \tag{38}$$

6. Efficiency of the angular momentum transport

If angular momentum is transported only by the internal waves we have considered here, the angular velocity evolves with time according to

$$\frac{\partial}{\partial t}\iint \Omega r^2 \sin^2\theta\, \rho r^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi = -\frac{\partial}{\partial r}\mathcal{L}_J(r) \tag{39}$$

or equivalently

$$\frac{\partial}{\partial t}(\rho r^4 \Omega) = \frac{1}{2}\frac{\rho_c v_c^3}{N_c \ell_c}\frac{\partial}{\partial r}\left[r^2\left(\frac{{\omega_c}^4}{I}\right)\frac{\Delta\Omega}{\omega_c}\right]. \tag{40}$$

To evaluate the efficiency of this transport, we neglect all variations except those of Ω; hence

$$\frac{\partial\Omega}{\partial t} \approx V_\mathrm{w}\frac{\partial\Omega}{\partial r}, \tag{41}$$

whose formal solution is

$$\Omega(r,t) = F\left(r + \int V_\mathrm{w}\,\mathrm{d}t\right). \tag{42}$$

Within this approximation the rotation profile, whose slope depends on the rate of angular momentum loss through the wind, propagates inwards with the velocity V_w, which is directly related to the damping integral $I(r)$ defined in (31):

$$V_\mathrm{w} = \frac{1}{2}\frac{\rho_c v_c^3}{\rho r^2}\frac{1}{N_c \omega_c \ell_c}\left(\frac{{\omega_c}^4}{I}\right). \tag{43}$$

Approximating $\rho_c v_c^3$ by 1/10 of the convective flux (see Cox & Giuli 1968), and this flux by $L_\odot/4\pi r_c^2$, we obtain a crude estimate for the synchronization time $t_\mathrm{sync} = r/V_\mathrm{w}$:

$$t_\mathrm{sync} \approx 60\,\frac{M_\odot R_\odot^2}{L_\odot}\,\frac{\rho}{\overline{\rho}}\left(\frac{r}{R_\odot}\right)^3\left(\frac{r_c}{R_\odot}\right)^2 N_c\omega_c\ell_c\left(\frac{I}{{\omega_c}^4}\right), \tag{44}$$

$\overline{\rho}$ being the mean density. The variation with depth of t_sync is depicted in Fig. 2; for this evaluation we took $\omega_c = 2\pi v_c/H_P$, $\ell_c = 2\pi r_c/H_P$, and assumed a penetration depth of $H_P/10$, which determines the value of N_c ($\approx 10^{-3}$ s^{-1}). The solar models were built with the stellar evolution code CESAM (Morel 1996).

The synchronization time depends somewhat on the age of the Sun, because the damping of the internal waves increases with time in the inner core, due to the steepening of the molecular weight gradient, but this effect is compensated by the increase of the convective frequency ω_c. We see that the internal waves are able to extract angular momentum from the solar interior in about 10^7 years, three orders of magnitude less than the present spin-down time.

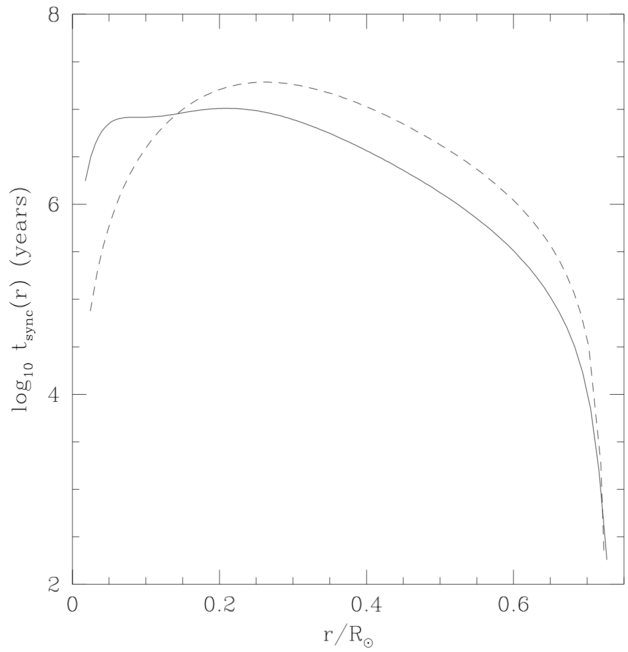

Fig. 2. Variation with depth of the synchronization time $t_{\rm sync} = r/V_w$ (see eq. 44). As in Fig. 1, the dashed line refers to the Sun at 200 Myr, the continuous line to the present Sun.

7. Conclusion

We conclude that the internal gravity waves are quite efficient in extracting angular momentum from the radiative interior of the Sun, a result anticipated by Schatzman (1993). During the later stages of the solar spindown, this process prevails over the other mechanisms which have been proposed so far, except perhaps magnetic torquing (see Introduction). The efficiency of this transport has been confirmed independently by Kumar & Quataert (1996), but these authors considered the differential damping between waves of opposite azimuthal order m, which redistributes angular momentum within the radiation zone with no net flux into the convective envelope. In contrast, we neglected here this effect altogether and focussed only on those waves which undergo complete damping at the critical level where their local frequency vanishes. We are implementing this transport in our rotational evolution codes, where it will compete with other mechanisms, such as meridian circulation, and we shall report the results in forthcoming papers.

But we clearly need to move beyond the present exploration, whose weak points ought to be recalled. First we have excluded the Coriolis force from the description of the internal waves, thus missing a possible unbalance between waves of opposite azimuthal order m. Second, we neglected the redistribution of angular momentum due to the differential damping mentioned above. Finally we used a rather crude recipe to couple the internal waves with the convective motions, which we assumed to obey Kolmogorov's law; it could well be that the stochastic excitation at wavelengths exceeding the size of the convective eddies is overestimated, but the result is not too sensitive to the slope of the power spectrum.

In spite of these imperfections, which have to be remedied, we see only one interpretation of the short characteristic time we have found above. Namely that the transport of angular momentum through the internal waves which are emitted by the convection zone is responsible for the flat rotation profile which is observed in the Sun.

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Appendix A: Expressions for the angular momentum flux

The angular momentum luminosity is given by eq. (29):

$$\mathcal{L}_J(r) = 4\pi r^2 \frac{\rho_c v_c^3}{N_c \ell_c}\, Q(r)$$

with

$$Q(r) = \int_{\omega_c}^{N_c} \frac{\mathrm{d}\omega}{\omega} \left(1 - \frac{\omega^2}{N_c^2}\right)^{\frac{1}{2}} \left[\frac{\omega}{\omega_c}\right]^{-3} \int_0^{\ell_u} \frac{\mathrm{d}\ell}{\ell} \int_0^{\ell} m\, \mathrm{d}m.$$

Let us recall that we have replaced the summations over integer m and ℓ by integrals over these quantities, which are assumed to vary continuously. We take $m = 1$ (and therefore $\ell = 1$) as lowest value when defining validity domains for various expressions. But we shall keep 0 as lower bound for the integrals, because for small values of L they approximate better sums such that

$$\sum_1^L \frac{1}{\ell} \sum_1^{\ell} m = \frac{1}{4} L^2.$$

We calculate first the angular momentum flux in the range

$$\ell_c^{-3} < \left(\frac{I}{\omega_c^4}\right) < 1$$

which corresponds to case II in Fig. 3; it covers most of the existence domain of the integral $I(r)$, as shown in Fig. 1. Then the upper limit of ℓ, for given ω, is fixed by condition (33)

$$\ell_u = \left(\frac{\omega^4}{I}\right)^{\frac{1}{3}}$$

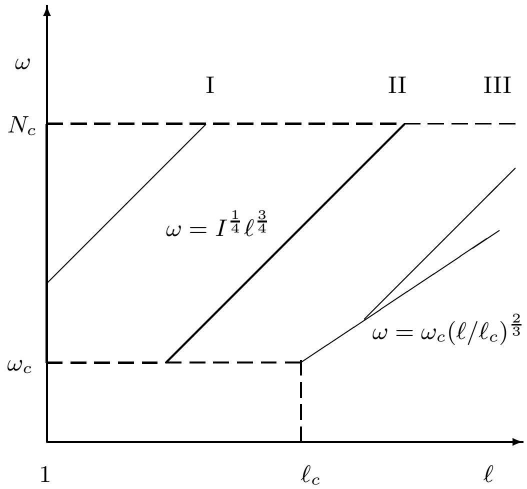

Fig. 3. Limits of the integration domain in the (ℓ,ω) plane, in logarithmic coordinates. In case II, which is the most relevant to the solar interior, the damping line $\omega = I^{1/4}\ell^{3/4}$ cuts $\omega = \omega_c$ between 1 and ℓ_c; the corresponding integration domain is delineated in thick lines.

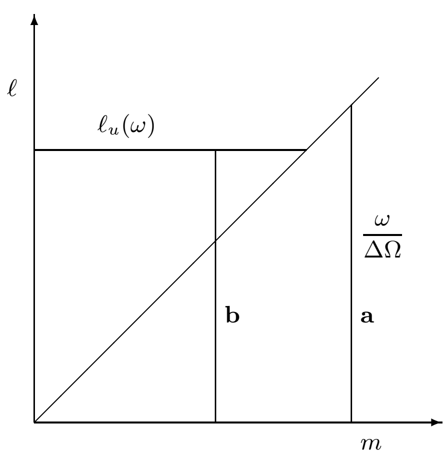

Fig. 4. Limits of the integration domain in the (m,ℓ) plane.

and the integration domain is split in two, according to whether ℓ_u is smaller or larger than $\omega/\Delta\Omega$ (see Fig. 4), or equivalently whether ω is smaller or larger than ω',

$$\text{with} \qquad \frac{\omega'}{\omega_c} = \frac{I}{\omega_c^4}\left(\frac{\omega_c}{\Delta\Omega}\right)^3.$$

Subdomain **a** : $\omega_c < \omega < \omega'$ (or $\ell_u < \omega/\Delta\Omega$)

In the (m,ℓ) plane the integration domain is a triangle:

$$P_a(\omega) = \int_0^{\ell_u}\frac{\mathrm{d}\ell}{\ell}\int_0^{\ell} m\,\mathrm{d}m = \frac{1}{4}\left(\frac{\omega_c^4}{I}\right)^{\frac{2}{3}}\left(\frac{\omega}{\omega_c}\right)^{\frac{8}{3}}$$

and therefore

$$\begin{aligned}
Q_a &= \int_{\omega_c}^{\omega'} P_a(\omega)\left(\frac{\omega}{\omega_c}\right)^{-3}\frac{\mathrm{d}\omega}{\omega} \\
&= \frac{1}{4}\left(\frac{\omega_c^4}{I}\right)^{\frac{2}{3}}\int_{\omega_c}^{\omega'}\left(\frac{\omega}{\omega_c}\right)^{-\frac{1}{3}}\frac{\mathrm{d}\omega}{\omega} \\
&= \frac{3}{4}\left(\frac{\omega_c^4}{I}\right)^{\frac{2}{3}}\left[1-\frac{\Delta\Omega}{\omega_c}\left(\frac{\omega_c^4}{I}\right)^{\frac{1}{3}}\right]. \qquad (45)
\end{aligned}$$

Subdomain **b** : $\omega' < \omega < N_c$ (or $\ell_u > \omega/\Delta\Omega$)

The integration domain in (m,ℓ) is now a trapezium; for the triangular tip we have

$$P_{b1}(\omega) = \int_0^{\omega/\Delta\Omega}\frac{\mathrm{d}\ell}{\ell}\int_0^{\ell} m\,\mathrm{d}m = \frac{1}{4}\left(\frac{\omega}{\Delta\Omega}\right)^2$$

and in the rectangular part

$$P_{b2}(\omega) = \int_{\omega/\Delta\Omega}^{\ell_u}\frac{\mathrm{d}\ell}{\ell}\int_0^{\omega/\Delta\Omega} m\,\mathrm{d}m = \frac{1}{2}\left(\frac{\omega}{\Delta\Omega}\right)^2\ln\left(\frac{\ell_u}{\omega/\Delta\Omega}\right).$$

Summing the two we get

$$\begin{aligned}
P_b = P_{b1}+P_{b2} &= \frac{1}{4}\left(\frac{\omega_c}{\Delta\Omega}\right)^2\left(\frac{\omega}{\omega_c}\right)^2 \\
&\times\left[1-\frac{2}{3}\ln\frac{I}{\omega_c^4}+2\ln\frac{\Delta\Omega}{\omega_c}+\frac{2}{3}\ln\frac{\omega}{\omega_c}\right].
\end{aligned}$$

It remains to integrate over ω; taking into account that $\omega_c \ll N_c$, we have

$$Q_b = \int_{\omega'}^{N_c} P_b(\omega)\left(\frac{\omega}{\omega_c}\right)^{-3}\frac{\mathrm{d}\omega}{\omega} = \frac{5}{12}\frac{\Delta\Omega}{\omega_c}\left(\frac{\omega_c^4}{I}\right). \qquad (46)$$

and hence

$$Q(r) = Q_a + Q_b = \frac{3}{4}\left(\frac{\omega_c^4}{I}\right)^{\frac{2}{3}} - \frac{1}{3}\left(\frac{\omega_c^4}{I}\right)\frac{\Delta\Omega}{\omega_c}, \qquad (47)$$

which is the result quoted above in eq. (37). This expression is valid provided $\omega_c < \omega' < N_c$, in other terms as long as

$$\frac{\omega_c}{N_c}\left(\frac{I}{{\omega_c}^4}\right) < \left(\frac{\Delta\Omega}{\omega_c}\right)^3 < \left(\frac{I}{{\omega_c}^4}\right). \qquad (48)$$

So far, we have implicitly assumed that $\omega_c < \omega' < N_c$; let us examine the other cases.

For $\omega' > N_c$, or

$$\left(\frac{\Delta\Omega}{\omega_c}\right)^3 < \frac{\omega_c}{N_c}\left(\frac{I}{{\omega_c}^4}\right) \qquad (49)$$

$Q_b = 0$ and $Q(r)$ does not depend on $\Delta\Omega$ anymore

$$Q(r) = Q_a = \frac{3}{4}\left(\frac{\omega_c^4}{I}\right)^{\frac{2}{3}}\left[1-\left(\frac{\omega_c}{N_c}\right)^{\frac{1}{3}}\right]. \qquad (50)$$

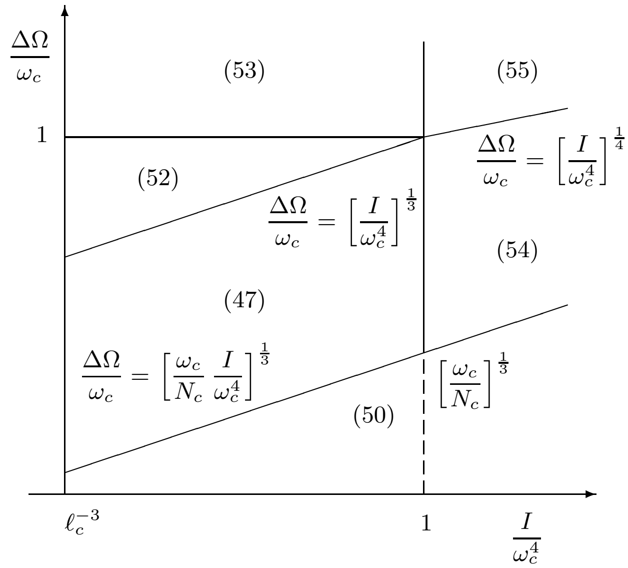

Fig. 5. The regions in the $(I/\omega_c^4, \Delta\Omega/\omega_c)$ plane where the angular momentum flux is given respectively by expressions (50), (47), (52), (53), (54) and (55). Logarithmic coordinates have been used again, but scales are arbitrary.

Last we turn to the case where $\omega' < \omega_c$, or

$$\left(\frac{I}{{\omega_c}^4}\right) < \left(\frac{\Delta\Omega}{\omega_c}\right)^3. \tag{51}$$

Then $Q_a = 0$, and the integration of P_b must be performed over the domain $\min(\omega_c, \Delta\Omega) < \omega < N_c$. The result is (again with $\omega_c \ll N_c$)

$$Q(r) = \left(\frac{\omega_c}{\Delta\Omega}\right)^2 \left[\frac{5}{12} - \frac{1}{6}\ln\frac{I}{\omega_c^4} + \frac{1}{2}\ln\frac{\Delta\Omega}{\omega_c}\right] \tag{52}$$

for $\Delta\Omega < \omega_c$, and

$$Q(r) = \left(\frac{\omega_c}{\Delta\Omega}\right)^3 \left[\frac{5}{12} - \frac{1}{6}\ln\frac{I}{\omega_c^4} + \frac{2}{3}\ln\frac{\Delta\Omega}{\omega_c}\right] \tag{53}$$

for $\Delta\Omega > \omega_c$.

Finally we have to consider the case I of Fig. 3, where $I > \omega_c^4$, which applies to the innermost core of the evolved Sun (see Fig. 1). Performing the integrations as above, one finds

$$Q(r) = Q_a + Q_b = \frac{3}{4}\left(\frac{\omega_c^4}{I}\right)^{\frac{3}{4}} - \frac{1}{3}\left(\frac{\omega_c^4}{I}\right)\frac{\Delta\Omega}{\omega_c}, \tag{54}$$

for $(\Delta\Omega/\omega_c)^4 < I/\omega_c^4$, and

$$Q(r) = \left(\frac{\omega_c^4}{I}\right)^{\frac{1}{4}} \left(\frac{\omega_c}{\Delta\Omega}\right)^2 \left[\frac{5}{12} - \frac{1}{8}\ln\frac{I}{\omega_c^4} + \frac{1}{2}\ln\frac{\Delta\Omega}{\omega_c}\right] \tag{55}$$

when $(\Delta\Omega/\omega_c)^4 > I/\omega_c^4$.

It is easy to check that $Q(r)$ given successively by (50), (47), (52), (53), (54) and (55) is a continuous function of $\Delta\Omega(r)$ and $I(r)$, and hence of r. In Fig. 5, we display the domains where each of these expressions applies.

Appendix B: Effect of a molecular weight gradient

A molecular weight gradient modifies somewhat the dynamics of internal waves, and in particular their radiative damping. To take this into account, we follow the method adopted by Press (1981), and refer to his equations as (P . .).

The equation of motion is still (P 27)

$$i\sigma\nabla^2(\rho u_r) = g k_h^2 \rho' \tag{56}$$

but the equation of state now includes the Eulerian perturbation of the molecular weight μ :

$$\frac{\rho'}{\rho} = -\delta\frac{T'}{T} + \varphi\frac{\mu'}{\mu}, \tag{57}$$

where we have ignored the pressure perturbation, as allowed for internal waves (anelastic approximation). The fluctuation of μ is given by

$$i\sigma\frac{\mu'}{\mu} = \frac{\nabla_\mu}{H_P}\,u_r, \tag{58}$$

which expresses the conservation of molecular weight in the wave motion.

To eliminate T', we call the heat equation

$$i\sigma\frac{T'}{T} = -(\nabla_{\rm ad} - \nabla)\,\frac{u_r}{H_P} + \frac{K}{T}\nabla^2 T', \tag{59}$$

K being the thermal diffusivity; using (57) and (58), and neglecting the vertical variation of all mean quantities compared to that of T', we get

$$i\sigma\frac{\rho'}{\rho} = \frac{N^2}{g}u_r - K\nabla^2\left(\delta\frac{T'}{T}\right), \tag{60}$$

where

$$N^2 = N_t^2 + N_\mu^2 = \frac{g\delta}{H_P}(\nabla_{\rm ad} - \nabla) + \frac{g\varphi}{H_P}\nabla_\mu.$$

Next we draw (T'/T) from the equation of state (57) to obtain

$$(i\sigma - K\nabla^2)\rho' = \frac{N^2}{g}\rho u_r - \frac{K}{i\sigma}\nabla^2\left(\frac{N_\mu^2}{g}\rho u_r\right) \tag{61}$$

which is (P 23) plus the extra term in N_μ^2. Combining (56) and (61), we are led to

$$\left(\nabla^2 + k_h^2\frac{N^2}{\sigma^2}\right)\rho u_r + i\frac{K}{\sigma}\nabla^2\left(\nabla^2 + k_h^2\frac{N_\mu^2}{\sigma^2}\right)\rho u_r = 0, \tag{62}$$

which again differs from (P 28) by this term in N_μ^2.

This equation is readily transformed into the dispersion relation

$$\left[k_v^2 \; - \; k_h^2 \left(\frac{N^2}{\sigma^2} - 1\right)\right]$$
$$- \; i \, \frac{K}{\sigma} \left[k_v^2 + k_h^2\right] \left[k_v^2 - k_h^2 \left(\frac{N_\mu^2}{\sigma^2} - 1\right)\right] = 0,$$

which yields the following expression for the vertical wavenumber, in the quasi-adiabatic limit:

$$k_v = \pm k_h \left(\frac{N^2}{\sigma^2} - 1\right)^{\frac{1}{2}} + i \, \frac{K}{2\sigma} \, k_h^3 \, \frac{N \, N_t^2}{\sigma^3} \left(\frac{N^2}{N^2 - \sigma^2}\right)^{\frac{1}{2}}, \tag{63}$$

and hence the value of the damping integral $\tau(r)$ quoted above in (19):

$$\tau(r) = [\ell(\ell+1)]^{3/2} \int_r^{r_c} K \, \frac{N \, N_t^2}{\sigma^4} \left(\frac{N^2}{N^2 - \sigma^2}\right)^{\frac{1}{2}} \frac{\mathrm{d}r}{r^3}.$$

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